# A CHARACTERIZATION OF CONVEXITY AND CENTRAL SYMMETRY FOR PLANAR POLYGONAL SETS†

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ABSTRACT

Let $S$ be a compact set in $R^2$ with nonempty interior, $L(u,k)$ be a line $\langle u, x\rangle = k$, and $\zeta_u(k)$ be the linear Lebesgue measure of $S \cap L(u, k)$. It is well known that for a convex $S$, $\zeta_u(k)$ is unimodal, that is, as a function of $k$, it is first nondecreasing and then nonincreasing for every $u \in R^2$. Further, if $S$ is centrally symmetric with respect to $M$, $\zeta_u(k)$ achieves maximum when $L(u, k)$ passes through $M$. Converse propositions are considered in this paper for polygonal $S$ with Jordan boundary. It is shown that unimodality alone does not suffice for convexity. However, if $\zeta_u(k)$ achieves maximum whenever $L(u, k)$ passes through some fixed point $M$ then unimodality yields convexity as well as central symmetry. It is also shown that continuity of $\zeta_u(k)$ in the interior of its support implies convexity of $S$. This last result, however, is false for non-polygonal sets.

## 1. Introduction and summary

Let $S$ be a compact set in the Euclidean plane $R^2$ having a non-empty interior. For a fixed non-zero vector $u \in R^2$ and $k \in R$, let $L(u, k)$ denote the line $u \cdot x = k$ and let $\phi_u(k)$ denote the linear Lebesgue measure of $S \cap L(u, k)$. If $S$ is a convex body then it is easy to verify that $\phi_u(k)$, as a function of $k$, is first non-decreasing and then non-increasing, no matter what $u$ is chosen. Moreover if the convex body $S$ is centrally symmetric with respect to a point $M$, then, for a fixed $u$, $\phi_u(k)$ achieves a maximum when $L(u, k)$ passes through $M$.

In this paper the converse propositions are considered. We prove (in a special case) that if $\phi_u(k)$ is non-decreasing first and then non-increasing and if $\phi_u(k)$

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achieves a maximum whenever $L(u, k)$ passes through some fixed point $M$, then $C$ is convex and centrally symmetric with respect to $M$. The special case considered here assumes that $S$ is a polygon whose boundary is a Jordan curve. It is believed however that the result holds for more general sets and that the proof may use some form of polygonal approximation.

## 2. Unimodal functions, mode and central symmetry

A non-negative function $f$ on $R$ is said to be *unimodal* if there is a $v \in R$ such that $f$ is non-decreasing on $(-\infty, v]$ and non-increasing on $[v, \infty)$. Such a number $v$ may not be unique.

Let $S$ be a compact set in $R^2$. For a non-zero $u \in R^2$ and $k \in R$, let $L(u, k)$ denote the line $u \cdot x = k$. Let $\phi_u(k)$ denote the linear Lebesgue measure of $S \cap L(u, k)$. The set $S \cap L(u, k_0)$ is said to be a *modal* (for the direction $u$) if

$$\phi_u(k_0) \geqq \phi_u(k) \quad \text{for all } k \in R.$$

A point $M \in R^2$ is called a *mode* of $S$ if every straight line passing through $M$ produces a modal intersection with $S$.

THEOREM 1. *Let $S$ be a compact convex body in $R^2$. Then* (i) *$S$ has at most one mode*; (ii) *if $S$ has a mode $M$, then $M$ is in the interior of $S$.*

PROOF. If possible, let $M_1$ and $M_2$ be the two modes of $S$. The separating hyperplane theorem shows that both $M_1$ and $M_2$ belong to $S$. Let $A_1M_1A_2$ and $B_1M_2B_2$ be the intercepts made on $S$ by straight lines perpendicular to $M_1 M_2$. (See Fig. 1.) Since $M_1$ and $M_2$ are both modes of $S$, the lengths of $A_1A_2$ and $B_1B_2$ must be equal. Therefore $A_1A_2B_2B_1$ is a parallelogram. Let $l_1$ and $l_2$, respectively denote the parallel straight lines determined by $A_1B_1$ and $A_2B_2$. Further, for $i = 1, 2$, let $P_iQ_i$ be the intercept made on $S$ by $l_i$. Since $S$ is convex and $M_1$ and $M_2$ are modes of $S$, it follows that $S$ lies entirely between the two lines $l_1$ and $l_2$. But this implies that $P_1Q_2$ and $P_2Q_1$ are unique modals (in their respective direc-

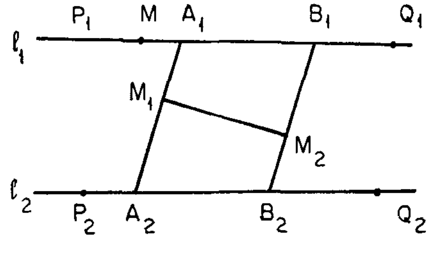


Fig. 1.

tions) and since these intersect in a single point, there cannot be two modes. This proves (i).

To prove (ii), assume that $S$ has a mode $M$. As remarked earlier $M$ cannot be outside $S$. If possible, let $M$ be on the boundary of $S$. Let $l_1$ be a line of support for $S$ through $M$. (See Fig. 1 again). Since $M$ is a mode, $S \cap l_1$ has a positive length. Let $P_1Q_1$ be this intersection. Let $l_2$ be the second line of support parallel to $l_1$ and let $P_2Q_2 = S \cap l_2$. Here $P_2$ and $Q_2$ may coincide. In any case, $P_1Q_2$ and $P_2Q_1$ are unique modals which do not intersect in $M$. This contradiction completes the proof of the theorem.

Hammer [1] proved the following result: If a convex body $S$ in $R^2$ possesses a mode $M$ in its interior, then $S$ is centrally symmetric with respect to $M$. The remark which follows his proof gives the impression that the assertion holds for unbounded $S$ also. However, a simple example shows that such a claim is false. Let $S$ be a closed strip bounded by two parallel straight lines. Clearly every point of $S$ is a mode; but $S$ is centrally symmetric only with respect to points on the line which is equidistant from the boundary lines.

Combining Hammer's result with our Theorem 1, we get the following theorem.

THEOREM 2. *If a compact convex body $S$ in $R^2$ has a mode $M$, then $S$ is centrally symmetric about $M$.*

### 3. Unimodality and convexity

Let $S$ be a compact set in $S^2$ having a non-empty interior. We will use the function $\phi_u(k)$ and the other terms defined in Section 2. Consider the following two conditions:

CONDITION A. For every fixed non-zero $u \in R^2$, the function $\phi_u(k)$ is unimodal in $k$.

CONDITION B. Besides satisfying condition A, set $S$ also possesses a mode $M$.

In what follows, the phrase "$S$ has a mode" will be considered equivalent to the phrase "$S$ satisfies condition B". This section tries to study the extent to which conditions A or B imply the convexity of $S$. To avoid trivial counterexample involving isolated points, lines protruding out of convex sets, etc., it is natural to assume that $S$ is the compact support of itself, in the sense that no proper compact subset of $S$ has the same planar Lebesgue measure as $S$. Even under this restriction condition A alone does not imply convexity; see Example 1 below. However, it is believed that condition B is sufficient to imply convexity. While we have been unable to settle this general problem, we are in a position to prove the following theorem. As already remarked in Section 1, the general case may follow by some kind of polygonal approximation.

THEOREM 3. *Let S be a compact polygon in $R^2$ with a Jordan boundary. If S has a mode M, then S is convex and centrally symmetric about M.*

The proof of this theorem is divided into several lemmas. Before we present these, we give some definitions and make some remarks. A set $S$ in a linear topological space is called *strongly locally convex* at a point $p \in S$ if there is a neighborhood $N$ of $p$ such that $N \cap S$ is convex. It is known (cf. Theorem 4.4 in Valentine [2]) that if a closed, connected set $S$ in a linear topological space is strongly locally convex at each of its points, then $S$ is convex. In particular, let $S$ be a compact polygon in $R^2$ with a Jordan boundary. Then it is clear that $S$ is strongly locally convex at each interior point of $S$ and also at each point on the boundary which is not a vertex. Our method of proving Theorem 3 therefore consists of proving that $S$ is strongly locally convex at each vertex. For simplicity in writing, we will call a vertex $A$ of $S$ convex if $S$ is strongly locally convex at $A$. The vertex will be called non-convex if it is not convex.

For later use, we state here an important property. Let $S$ be a compact polygon in $R^2$ with a Jordan boundary. Let $B$ be a non-convex vertex of $S$ formed by the sides $AB$ and $BC$. (See Fig. 2; the shaded portion is a part of $S$.) Let $l$ be the line determined by $BC$. Then $B$ can be approached by points outside $S$ only from that side of $l$ on which $A$ lies. In fact such a property characterizes non-convex vertices of $S$.

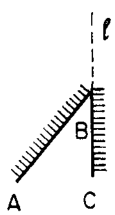

Fig. 2

EXAMPLE 1. It was remarked that condition $A$ alone does not imply convexity, even for polygons with Jordan boundary. To see this, let $ABCD$ and $A'B'C'D'$ be parallelograms which are mirror images of each other and are such that $CD$ and $C'D'$ meet outside the polygon $ABCDD'C'B'A'$. This polygon can be seen to satisfy condition A but is not even star-shaped. (We are grateful to George Converse for modifying our original example where $S$ was a star-shaped nonconvex polygon.)

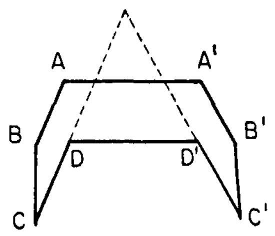


Fig. 3

We now proceed to present the lemmas which will imply Theorem 3. It is to be understood that $S$ is a compact polygon with a Jordan boundary. It is also to be understood that $S$ has a finite number of vertices. We use the symbol $S^*$ to denote the kernel of $S$ (i.e., the set of points with respect to which $S$ is star-shaped).

LEMMA 1. *Let $S$ have a mode $M$ which is in the interior of $S^*$. Then $S$ is convex and centrally symmetric with respect to $M$.*

PROOF. Let $V$ be a neighborhood of $M$ such that $V \subset S^*$. Let $K$ be a boundary point of $S$. Then there cannot be any point $K'$ of $S$ on the extension of $MK$; for otherwise, $K$ would become an interior point of $S$. (See Fig. 4.) Thus a side passing through $K$ cannot lie along the line determined by $MK$. Further, if $K$ is a vertex, then the two sides $KJ$ and $KL$ (say) of $S$ through $K$ lie on opposite sides of the ne determined by $MK$. This fact will be used below.

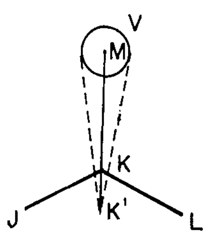


Fig. 4

If possible, let $S$ have a non-convex vertex $B$. Extend $BM$ until it comes out of $S$ at $B'$. (See Fig. 5.) If $B'$ is not a vertex, then it is easy to see that $BMB'$ is not modal. Therefore $B'$ must be a vertex. Let $A'B'$ and $B'C'$ be the sides passing through $B'$ and $AB$ and $BC$ those passing through $B$ with $A'$ and $A$ on the same side of the line $BMB'$. This side of $BMB'$ will be called as "left side of $BMB'$."

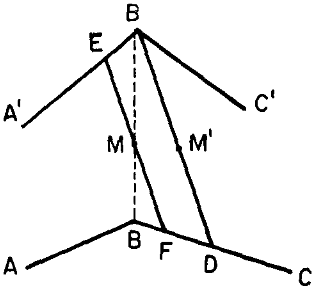


Fig. 5

Now $B'A'$ and $CB$ must meet on the left side of $BMB'$; for, otherwise, the intercept parallel to $BMB'$ to the left of $BMB'$ would be larger than $BMB'$. Now choose $M'$ to the right of $BMB'$ and so close to $M$ that $M' \in S^*$ and $B'M'$ when extended leaves $S$ at a point $D$ which is on the side $BC$. Now the line through $M$ parallel to $B'M'D$ leaves an intercept $EMF$ on $S$ such that $EMF < B'M'D$. This contradiction proves that $S$ does not have a non-convex vertex. This proves that $S$ is convex. The central symmetry follows from Theorem 2. The proof of the lemma is thus complete.

It is evident from Lemma 1 that Theorem 3 would be proved as soon as we prove that

$$S \text{ has a mode } M \Rightarrow M \text{ is in the interior of } S^*.$$

This will be done through a series of steps. The next two lemmas will be used in the proofs of subsequent lemmas.

LEMMA 2. *If a point $K$ is such that $AK \subset S$ for every vertex $A$ of $S$, then $K \in S^*$.*

PROOF. Let $AK \subset S$ for every vertex $A$ of $S$. If possible, let $K$ be outside $S^*$. Then there exists a point $C$ on the boundary of $S$ such that $CK$ is not entirely in $S$. Let $AB$ be the side containing $C$. Then the sides of the triangle $ABK$ are all in $S$, whereas there is a point $x$ on $CK$ such that $x \in S^c$, the complement of $S$. Thus there are points of $S^c$ both within and outside the triangle $ABK$. But then $S$ cannot have a Jordan boundary. This contradiction proves the lemma.

EXAMPLE 2. Lemma 2 fails if the boundary of $S$ is not assumed to be a Jordan curve. For example, let $ABC$ be a triangle and $D$ and $E$ be two points in the interior of $ABC$. Let $S$ be the set of points of $ABC$ which are not in the interior of triangle $ADE$. The vertex $A$ satisfies the hypothesis of Lemma 2 but clearly, $S$ is not star-shaped.

LEMMA 3. *Let $\xi$ be outside $S^*$. Then there exists a point $x_0 \in S$ such that*: (i) *$x_0$ is a boundary point of $S$*; (ii) *$x_0$ is not a vertex of $S$*; (iii) *the side $l$ of $S$ containing $x_0$ does not pass through $\xi$ when extended in either direction*; (iv) *there is a point $y_0$ on the line segment $x_0\xi$ such that $y_0 \neq x_0$ and the entire open line segment $(y_0, x_0)$ is outside $S$.*

PROOF. Since $\xi$ is outside $S^*$, there exists a point $z \in S$ such that the line segment $\xi z$ is not entirely in $S$. If we proceed from $z$ to $\xi$ in a straight line, then we must leave $S$ for the first time at $x$ (say) and then re-enter $S$ at $y$ (say). Here $x$ and $z$ could coincide. Also $y$ and $\xi$ could be the same. If, after leaving $S$ at $x$, we do not meet $S$ again before reaching $\xi$, then we take $y = \xi$. Two cases arise.

*Case* 1. Let $x$ not be a vertex of $S$ and let $l$ be the side containing $x$. (See Fig. 6.) Then $l$, when extended, cannot pass through $\xi$, for, otherwise, while proceeding from $z$ to $\xi$, we would not leave $S$ at $x$. Taking $x_0 = x$ and $y_0 = y$ we see that all the conclusions of the lemma are satisfied.

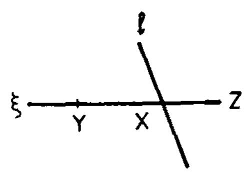


Fig. 6

*Case* 2. Let $x$ be a vertex of $S$ and let $xA, xB$ be the sides meeting at $x$. Relabelling the vertices if necessary we may assume that the angle $\xi xA$ is less than 180° and is not greater than the angle $\xi xB$. (Various positions of $xB$ are shown by dotted lines in Fig. 7. The angle $\xi xA$ could be acute.) Let $y'$ be in the open line segment $yx$. Since $y' \in S^c$, there is a circular neighborhood $N$ of $y'$ such that $N \subset S^c$. Choose $x'$ in the open line segment $xA$ so close to $x$ that $\xi x'$ intersects $N$ in an interval. We can now take $x'$ as the $x$ of Case 1 above. The proof of the lemma is thus complete.

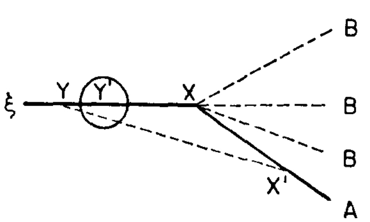


Fig. 7

LEMMA 4. *If $S$ has a mode $M$, then $S$ is star-shaped with respect to $M$.*

PROOF. Suppose $S$ is not star-shaped with respect to $M$. Then there exist points $x_0, y_0$ and a side $l$ satisfying the conclusions of Lemma 3. Let the equation of the line determined by $l$ be $u \cdot x = c_0$. Further let $u \cdot M < c_0$. Choose $c_1 < c_0$ but close to $c_0$ so that $u \cdot y_0 < c_1$ and there is no vertex of $S$ in the region $c_1 \leqq u \cdot x < c_0$. Let $T = \{x \in S \mid c_1 \leqq u \cdot x < c_0\}$ and $T_c = \{x \in S \mid u \cdot x = c\}$. The set $T$ must be non-empty; for, otherwise $S$ would not be connected. Since there is no vertex of $S$ in $T$, it follows (see Fig. 8) that $T$ is a finite disjoint union of trapezia bounded

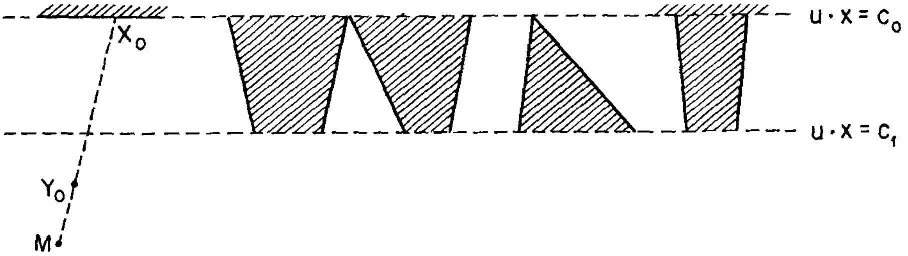


Fig. 8

by the lines $u \cdot x = c_0$ and $u \cdot x = c_1$. We note the following properties of the trapezia : (i) the trapezia are open from the top side; (ii) a trapezium may degenerate into a triangle; (iii) two trapezia may have a common limiting point on $u \cdot x = c_0$. Now let $G = T_{c_0} \cap \bar{T}$ where $\bar{T}$ is the closure of $T$, and $G' = T_{c_0} - G$. Let $\lambda$ denote linear Lebesgue measure. From the choice of $x_0, y_0$ and $l$, it is clear that, with the exception of the end points, the entire side $l$ is a subset of $G'$. Therefore $\lambda(G') > 0$. It is easy to see that $G$ consists precisely of the top sides of the trapezia which constitute $T$. It follows that $\lambda(T_c) \to \lambda(G)$ as $c \to c_0$ from below. (Another way to see this is that, for each trapezium, we are only measuring the difference between two linear functions. We must therefore have continuity.) Now since $G \cap G' = \phi$,

$$\lambda(T_{c_0}) = \lambda(G \cup G') = \lambda(G) + \lambda(G') > \lambda(G) = \lim_{c \to c_0} \lambda(T_c).$$

Therefore, if $c$ is sufficiently close to $c_0$ and $c < c_0$, then $\lambda(T_c) < \lambda(T_{c_0})$. Hence $S$ cannot have a mode in the region $u \cdot x < c_0$. In particular $M$ cannot be a mode of $S$. This completes the proof of the lemma.

LEMMA 5. *Let $S$ have a mode $M$. Then $M$ is in the interior of $S$. Moreover $M$ cannot lie on any line determined by a side of $S$ passing through a non-convex vertex of $S$.*

PROOF. By Lemma 4, $M \in S^*$. Therefore $M \in S$. It is easy to see that $M$ cannot be a convex vertex of $S$ nor can it be a boundary point which is not a vertex, since otherwise every vertex of $S$ must be convex, so $S$ is convex (and we have the situation of Theorem 1). Thus the lemma will follow as soon as the second assertion is proved. So, let $C$ be a non-convex vertex of $S$ and let $BC$ and $CD$ be the sides meeting at $C$. Let $l$ be the line determined by $CD$. If possible let $M \in l$. Since $M \in S^*$, it is easy to see that by extending $DC$ a boundary point $P$ of $S$ could be found from which a side (or a part of a side) $PQ$ comes out such that $Q$ is on the opposite side of $l$ from $B$. (See Fig. 9.) Further $M$ has to be in the

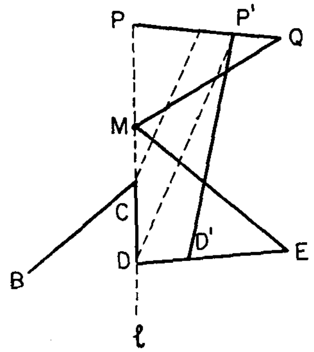


Fig. 9

closure of the line segment $CP$. Although the assumption of Jordan boundary prohibits $C$ and $P$ to coincide, $M$ could be either at $C$ or $P$. Again $M \in S^*$ implies that the side $DE$ through $D$ must come out on the opposite side of $l$ from $B$. Choose points $P'$ and $D'$ on $PQ$ and $DE$ respectively. If possible let $P'D' \subset S$ Then the entire quadrilateral $PP'D'D$ is a subset of $S$. But then $P'D$ is larger than the parallel intercept through $C$. This contradicts the fact that $M$ is a mode. Therefore $P'D'$ must have a point outside $S$. Since the triangles $PQM$ and $DEM$ are subsets of $S$, we see, by letting $P' \to P$ and $D' \to D$ that $M$ is a boundary point of $S$ which can be approached by points in $S^c$ from the opposite side of $l$ from $B$. But then $M$ cannot be at $C$. Thus $M$ cannot be at any non-convex vertex. But since $M$ is a boundary point, we get a contradiction. This proves that $M$ is outside $l$ and completes the proof of the lemma.

LEMMA 6. *Let $S$ have a mode $M$. Then $M$ is in the interior of $S^*$.*

PROOF. By Lemma 5, $M$ is in the interior of $S$. Let $V$ be a circular neighborhood of $M$ such that $V \subset S$. Let $B$ be a vertex of $S$. Then if a side of $S$ starting from $B$ lies along the line $l$ determined by $MB$ then, $M$ being in $S^*$, there would be a concave vertex on $l$, which would contradict Lemma 5. Further the two sides starting from $B$ cannot come out on the same side of $l$; for, otherwise, $S$ would not be star-shaped with respect to $M$. Thus the situation is as shown in Fig. 10, where $AB$ and $BC$ are the sides meeting at $B$. Now the union of $V$, the

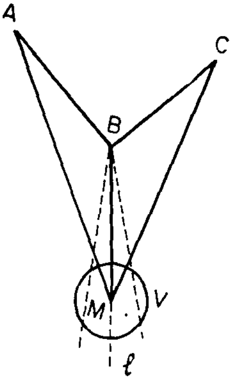


Fig. 10

triangle $ABM$ and the triangle $BCM$ is a subset of $S$. It is then clear from the figure that there is a neighborhood $V_B$ of $M$ such that for every $x \in V_B$, the line segment $xB \subset S$. Since the number of vertices is finite, we can find a single neighborhood $W$ of $M$ such that for every $x \in W$ and for every vertex $y$ of $S$, the line segment $xy \subset S$. Lemma 2 now shows that $W \subset S^*$. This completes the proof of the lemma.

In view of Lemma 1 and 6, we see that the proof of Theorem 3 is complete.

## 4. Strong unimodality and convexity

We remarked in the introduction that a convex body in $R^2$ satisfies condition A (i.e. the function $\phi_u(k)$ is unimodal in $k$ for each fixed $u$). We also showed in Section 3 (see Fig. 3) that condition A alone does not imply convexity even for polygons with Jordan boundaries. But a convex body actually satisfies a condition stronger than condition A, namely, the function $\phi_u(k)$ is concave on its support, for every fixed $u$. This implies that, for a compact convex body, the function $\phi_u(k)$ is continuous on the interior of its support. We shall show that this last condition is strong enough to imply convexity for compact polygons with Jordan boundaries.

THEOREM 4. *Let $S$ be a compact polygon in $R^2$ with a Jordan boundary. If the function $\phi_u(k)$ is continuous on the interior of its support, for every fixed $u$, then $S$ is convex.*

PROOF. Suppose $S$ is not convex. Then there is a $\xi \in S$ such that $S$ is not star-shaped with respect to $\xi$. But then the proof of Lemma 4 shows that there is a $u$ such that $\phi_u(k)$ is discontinuous in the interior of its support. This contradiction proves the theorem.

It should be noted that condition A is not used in Theorem 4. This theorem is, of course, false if $S$ is not a polygon with a Jordan boundary. It is natural to ask whether the continuity of $\phi_u(k)$ and condition A would imply the convexity of general planar sets with Jordan boundaries. We are grateful to the referee for providing a negative answer to this question. As a counterexample, one can take the crescent formed by points which are inside the circle $x^2 + y^2 = 9$ and outside the circle $x^2 + (y-4)^2 = 25$.

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